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Note On Orthonormal Expansions

Communicated by Oved Shisha

THEOREM. *Let* $-\infty < a < b < \infty$ *and let* $\omega(x)$ *be a real function, summable over* (a, b) *and* $\geqslant 0$ *there. Let* $g_1, g_2, \ldots, g_n$ *be real functions, continuous on* $[a, b]$ *and orthonormal there with respect to the weight* ω. *Then, for any reals* $a_1, a_2, \ldots, a_n$, *we have*

$$\sum_{k=1}^{n} |a_k| \leqslant \left[n \int_a^b \omega(x)\, dx\right]^{1/2} \cdot \max_{a \leqslant x \leqslant b} \left|\sum_{k=1}^{n} a_k g_k(x)\right|. \tag{1}$$

If f *is a real function, continuous on* $[a, b]$, *and if* a_k *is its Fourier coefficient with respect to* g_k *and to the weight* ω:

$$a_k = \int_a^b \omega(x) f(x) g_k(x)\, dx, \qquad k = 1, 2, \ldots, n, \tag{2}$$

then

$$\sum_{k=1}^{n} |a_k| \leqslant \left[n \int_a^b \omega(x)\, dx\right]^{1/2} \cdot \max_{a \leqslant x \leqslant b} |f(x)|. \tag{3}$$

Proof. Let f be a real function, continuous on $[a, b]$, and let (2) hold. Since $\sum_{k=1}^{n} a_k^2 \leqslant \int_a^b \omega(x) f^2(x)\, dx$, we have by the Cauchy–Schwarz inequality,

$$\sum_{k=1}^{n} |a_k| = \sum_{k=1}^{n} |a_k| \cdot 1 \leqslant \left[\sum_{k=1}^{n} a_k^2 \sum_{k=1}^{n} 1^2\right]^{1/2} \leqslant n^{1/2} \left[\int_a^b \omega(x) f^2(x)\, dx\right]^{1/2}$$

$$\leqslant \left[n \int_a^b \omega(x)\, dx\right]^{1/2} \cdot \max_{a \leqslant x \leqslant b} |f(x)|.$$

Let now $a_1, a_2, \ldots, a_n$ be arbitrary reals. Then, clearly, (2) holds for $f(x) \equiv \sum_{j=1}^{n} a_j g_j(x)$. Hence, by (3), we have (1).

EXAMPLE. Consider the functions

$$\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}}\cos x, \frac{1}{\sqrt{\pi}}\sin x, \frac{1}{\sqrt{\pi}}\cos 2x,$$
$$\frac{1}{\sqrt{\pi}}\sin 2x,..., \frac{1}{\sqrt{\pi}}\cos nx, \frac{1}{\sqrt{\pi}}\sin nx \qquad (n \geqslant 1), \tag{4}$$

orthonormal on $[-\pi, \pi]$. By (1), for any reals $a_0, a_1, b_1, ..., a_n, b_n$,

$$| a_0 | + \sum_{k=1}^{n} | a_k | + | b_k |$$
$$\leqslant [2(2n+1)]^{1/2} \cdot \max_{-\pi \leqslant x \leqslant \pi} \left| \frac{a_0}{\sqrt{2}} + \sum_{k=1}^{n} a_k \cos kx + b_k \sin kx \right|.$$

Let f be a real function, continuous on $(-\infty, \infty)$ and of period 2π there, let its ordinary Fourier series be $(\alpha_0/2) + \sum_{k=1}^{\infty} \alpha_k \cos kx + \beta_k \sin kx$, and let $a_0, a_1, b_1, ..., a_n, b_n$, respectively, be its Fourier coefficients with respect to the functions (4) and the weight 1. Then

$$a_0 = \sqrt{\frac{\pi}{2}}\,\alpha_0; \qquad a_k = \sqrt{\pi}\,\alpha_k, \qquad b_k = \sqrt{\pi}\,\beta_k, \qquad k = 1, 2, ..., n.$$

Since, by (3), $| a_0 | + \sum_{k=1}^{n} | a_k | + | b_k | \leqslant [2\pi(2n+1)]^{1/2} \cdot \max_{-\pi \leqslant x \leqslant \pi} |f(x)|$, we have

$$\frac{| \alpha_0 |}{\sqrt{2}} + \sum_{k=1}^{n} | \alpha_k | + | \beta_k | \leqslant [2(2n+1)]^{1/2} \cdot \max_{-\pi \leqslant x \leqslant \pi} |f(x)|.$$

This note corrects an error in [1], Problem 21, p. 125.

REFERENCE

1. E. W. CHENEY, "Introduction to Approximation Theory," McGraw-Hill, New York, 1966.

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